Worcester County Mathematics League

FRESHMAN Meet 3
April 4, 2012

Coaches' Copy
Rounds, Answers and Solutions

Worcester County Mathematics League

Freshman Meet 3 - April 4, 2012

Round 1: Graphing on a Number Line

NO CALCULATOR ALLOWED

1

Draw the graph of each of the following inequalities described below on the corresponding number line provided at the bottom of the page. Please specify all endpoints on your graph.

1. I'm thinking of a number. Three more than twice my number is less than 17, but at least -1. Graph the possible values of the number that I am thinking of.

2. $\frac{x}{8} - \frac{x-2}{3} \geq \frac{x+1}{6} - 1$

3. $|x-10| < |x-2|$

ANSWERS

(1 pt.) 1. ←——————————→

(2 pts.) 2. ←——————————→

(3 pts.) 3. ←——————————→

Westborough, Quaboag, Worcester Academy

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Round 2: Operations on Polynomials

All answers must be in simplest exact form

NO CALCULATOR ALLOWED

2

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1. Simplify the following polynomial expression:

$$\left(2xy^2z\right)\left(5x^2y^2z^2\right)+y\left(-xyz\right)\left(-6x^2y^2z^2\right)$$

2. Factor the following polynomial as the product of two trinomials:

$$m^2+9n^2-81+6mn$$

3. Consider the polynomial $P(x)=x^3-9ax^2+9a^2x+40a^3$, where a is a constant. In terms of a, find the remainder when $P(x)$ is divided by $x-3a$.

ANSWERS

(1 pt.) 1.____________________

(2 pts.) 2.____________________

(3 pts.) 3.____________________

Millbury, Sutton, Worcester Academy

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Round 3: Techniques of Counting and Probability

3

All answers must be in simplest exact form

NO CALCULATOR ALLOWED

1. While cleaning out her garage, Ms. O'Garr found four old single-digit house numbers: one 3, one 4, and two 5's. Find the number of different two-digit house numbers she could create using any two of them.

2. An urn contains 12 white marbles labeled with the letters A through L, and 8 red marbles labeled with the letters A through H. Find the number of different ways that 3 white marbles and 2 red marbles could be chosen from the urn. Assume that the order that the marbles are chosen does not matter.

3. Sixteen index cards all have different numbers written on them; twelve of the numbers are even and the others are odd. If two of the cards are chosen at random, compute the probability that the sum of their numbers is even.

ANSWERS

(1 pt.) 1.____________________

(2 pts.) 2.____________________

(3 pts.) 3.____________________

Hudson, Algonquin, Auburn

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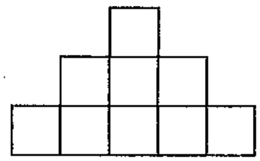
Round 4: Perimeter, Area and Volume

All answers must be in simplest exact form

4

NO CALCULATOR ALLOWED & THE DIAGRAM IS NOT DRAWN TO SCALE

1. The figure below shows nine congruent squares. If the total area of the figure is 144 square inches, compute the perimeter of the entire figure (in inches).



2. A rectangular garden is 20 feet long and 10 feet wide. The garden is entirely surrounded by a rectangular grass border that is 5 feet wide. The grass border is completely surrounded by a rectangular gravel pathway that is also 5 feet wide. The gravel pathway is surrounded by another rectangular grass border that is 5 feet wide. Compute the total area that is covered by grass (in square feet).

3. The sum of the lengths of all twelve edges of a rectangular solid is 20 centimeters. The distance from one vertex of the solid to the opposite vertex furthest away is 4 centimeters. Compute the total surface area of the solid (in square centimeters).

ANSWERS

(1 pt.) 1.______________________ inches

(2 pts.) 2.______________________ square feet

(3 pts.) 3.______________________ square centimeters

Millbury, Worcester Academy, QSC

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TEAM ROUND

All answers must *either* be in simplest exact form *or* as decimals rounded correctly to at least three decimal places! (3 pts. each)

APPROVED CALCULATORS ALLOWED

1. Mr. Taylor would take 5 days to paint a house working alone. He worked alone for 2 days and then was joined by Mr. Gregory and together they finished painting the house in 1 day. On the second painting job, working together they finished in 8 days. How many days would it have taken Mr. Taylor working alone to complete the second job?

2. Factor the following polynomial as the product of three binomials:
$$5x^3 - 6y^3 + 6x^2y - 5xy^2$$

3. The inside roof and inside supporting walls of rectangular tunnel that is 110 feet long must be covered with aluminum sheathing. The sheathing is available in long strips that are 33 inches wide and they cannot be cut narrower. The tunnel spans 28 feet across and stands 15 feet high. Compute the number of feet of aluminum sheathing that will be required to cover the roof and walls of the tunnel. Assume that there will be no overlap among the strips of sheathing and that each strip goes from the bottom of one wall to the top, across the roof, and down the other wall.

4. The points (2, 7) and (6, 3) are equidistant from the line $y = kx$. Compute the positive value of k.

5. There are two seven-digit numbers 1,084,AB8 that are a multiple of 88. Compute both ordered pairs of digits (A, B).

6. On the space provided on the answer sheet, graph the solution set of:
$$|x+3| + |x+4| < 7$$

7. There are five exterior doors to an office building. The doors are labeled 1, 2, 3, 4, and 5. Upon one trip in and out of the building, if a person must enter and leave through different doors, compute the probability that at least one of the doors used will have an even number. Assume that the person chooses the doors at random and is equally likely to choose one door over another.

8. How many three-digit numbers have three distinct digits in increasing or decreasing order? Only consider numbers whose leading digit is not zero.

St. John's, Bancroft, Westborough, Auburn, Hudson, Westborough, Tahanto, Assabet Valley

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ANSWER SHEET – TEAM ROUND

All answers must *either* be in simplest exact form *or* as decimals rounded correctly to at least three decimal places! (3 pts. each)

1.______________________________ days

2.______________________________

3.______________________________ feet

4.______________________________

5. (______, ________) and (________, ________)

6. ←――――――――――――――――――――→

7.______________________________

8.______________________________

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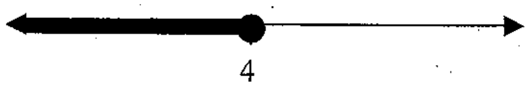
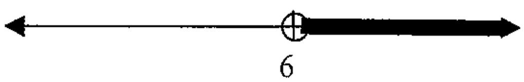
ANSWERS

Round 1

1. 

2. 

3. 

Round 2

1. $16x^3y^4z^3$

2. $(m+3n+9)(m+3n-9)$
(or equivalent, allowing for the commutativity of multiplication & addition)

3. $13a^3$

Round 3

1. 7

2. 6160

3. $\frac{3}{5} = 0.6 = 60\%$

Round 4

1. 64

2. 1200

3. 9

Team Round

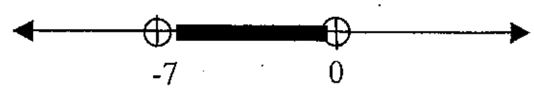
1. 24

2. $(5x+6y)(x+y)(x-y)$
(or equivalent, allowing for the commutativity of multiplication & addition)

3. 2320

4. $\frac{5}{4} = 1\frac{1}{4} = 1.25$

5. (2, 4) and (6, 8)
(the pairs can be written in either order)

6. 

7. $\frac{7}{10} = 0.7 = 70\%$

8. 204

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SOLUTIONS

Round 1

1. Let $x =$ the number that I am thinking of so that $-1 \le 2x + 3 < 17 \Rightarrow -4 \le 2x < 14 \Rightarrow -2 \le x < 7$.

2. Multiply by 24 then solve: $\frac{x}{8} - \frac{x-2}{3} \ge \frac{x+1}{6} - 1 \Rightarrow 3x - 8x + 16 \ge 4x + 4 - 24 \Rightarrow -9x \ge -36 \Rightarrow x \le 4$.

3. When $x \le 2$, we have $-x + 10 < -x + 2 \Rightarrow$ impossible. Next, when $2 < x < 10$, we have $-x + 10 < x - 2 \Rightarrow -x + 10 < x - 2 \Rightarrow 2x > 12 \Rightarrow x > 6$. Finally, when $x \ge 10$, we have $x - 10 < x - 2 \Rightarrow$ impossible. Therefore, the solution set is $x > 6$. Alternatively, the inequality asks: "What numbers are farther from 2 than they are from 10?" Six is the same distance from 2 and 10, and any number to the right of 6 is farther from 2 than it is from 10.

Round 2

1. $\left(2xy^2z\right)\left(5x^2y^2z^2\right) + y\left(-xyz\right)\left(-6x^2y^2z^2\right) = 10x^3y^4z^3 + 6x^3y^4z^2 = 16x^3y^4z^3$.

2. Factor, first by grouping to get a perfect square trinomial, then by the difference of two squares: $m^2 + 9n^2 - 81 + 6mn = m^2 + 6mn + 9n^2 - 81 = (m+3n)^2 - 81 = (m + 3n - 9)(m + 3n + 9)$.

3. The Remainder Theorem gives: $27a^3 - 81a^3 + 27a^3 + 40a^3 = 13a^3$.

Round 3

1. The two-digit house numbers that can be made are: 34, 35, 43, 45, 53, 54, and 55 $\Rightarrow$ there are 7.

2. There are ${}_{12}C_3 = \frac{12!}{3!\cdot 9!} = \frac{12\cdot 11\cdot 10}{3\cdot 2\cdot 1} = 220$ ways of selecting the three white marbles. There are ${}_{8}C_2 = \frac{8!}{2!\cdot 6!} = \frac{8\cdot 7}{2\cdot 1} = 28$ ways of selecting the two red marbles. Hence, by the fundamental counting principle, there are $220 \cdot 28 = 6{,}160$ ways to select all 5 marbles.

3. The sum of the numbers on the two cards will be even iff both cards are even or both cards are odd. The probability of selecting two even cards is $\frac{12}{16}\cdot\frac{11}{15}$ and the probability of selecting two odd cards is $\frac{4}{16}\cdot\frac{3}{15}$. Therefore, the probability that the sum of numbers is even is $\frac{12}{16}\cdot\frac{11}{15} + \frac{4}{16}\cdot\frac{3}{15} = \frac{132+12}{16\cdot 15} = \frac{144}{16\cdot 15} = \frac{9}{15} = \frac{3}{5}$.

Round 4

1. There are 9 squares having a total are of 144 $\Rightarrow$ the area of each square is 16 $\Rightarrow$ the side length of each square is 4. Therefore, the perimeter of the entire figure is $16 \cdot 4 = 64$.

2. The outer grass strip is a 50 by 40 rectangle, minus a 40 by 30 rectangle $\Rightarrow 50\cdot 40 - 40\cdot 30 = 800$. The inner grass border is a 30 by 20 rectangle minus a 20 by 10 rectangle $\Rightarrow 30\cdot 20 - 20\cdot 10 = 400 \Rightarrow$ the total area of grass is $800 + 400 = 1200$.

3. Let a, b, c = the length, width and height of the solid so that $4a + 4b + 4c = 20 \Rightarrow a + b + c = 5$. Next, the length of the solid's main diagonal is $\sqrt{a^2 + b^2 + c^2} = 4 \Rightarrow a^2 + b^2 + c^2 = 16$. Now, notice that

$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$, where $2ab+2ac+2bc = SA$, the solid's surface area. Therefore, by substitution, we have $5^2 = 16+SA \Rightarrow SA = 9$.

Team Round

1. Let $g =$ the number of days it would take Mr. Gregory to finish the first job working alone. Then, $\frac{2}{5}+\frac{1}{5}+\frac{1}{g} = 1 \Rightarrow g = \frac{5}{2} \Rightarrow$ Mr. Gregory works twice as fast as Mr. Taylor. Now, let $t =$ the number of days it would take Mr. Taylor to complete the second job alone so that $\frac{t}{2} =$ the # of days it would take Mr. Gregory to complete the second job alone. We have: $\frac{8}{t}+\frac{8}{\frac{t}{2}} = 1 \Rightarrow \frac{8}{t}+\frac{16}{t} = 1 \Rightarrow t = 24$.

2. Grouping and the difference of 2 squares yields $5x^3-5xy^2+6x^2y-6y^3 = 5x(x^2-y^2)+6y(x^2-y^2)$ $=(5x+6y)(x-y)(x+y)$.

3. 33 inches = 2.75 feet so that $110 \div 2.75 = 40 \Rightarrow 40$ strips of sheathing are needed. Each of the 40 strips must be $15+28+15 = 58$ feet long. Therefore, it will take $40 \cdot 58 = 2{,}320$ feet of sheathing.

4. The line $y = kx$ will pass through the midpoint of the segment whose endpoints are (2, 7) and (6, 3). The midpoint is (4, 5). Therefore, the line $y = kx$ has slope $k = \frac{5}{4}$.

5. In order to be divisible by 88, the number must be divisible by 11. If the number is divisible by 11, it must be true that $1-0+8-4+A-B+8 = 13+(A-B)$ is a multiple of 11 $\Rightarrow A-B = -2 \Rightarrow B-A = 2$. Next, in order for the number to be divisible by 88, it must be divisible by 8. If it is divisible by 8, then AB8 must be a three-digit number that it divisible by 8. Checking three-digit numbers that have $B-A = 2$ and that are divisible by 8 yields $(A,B) = (2,4)$ and $(6,8)$.

6. When $x < -4$, $|x+3|+|x+4| < 7 \Rightarrow -(x+3)-(x+4) < 7 \Rightarrow -2x-7 < 7 \Rightarrow x > -7$ so that $-7 < x < -4$. Next, when $-4 \le x \le -3$, $|x+3|+|x+4| < 7 \Rightarrow -(x+3)+(x+4) < 7 \Rightarrow 1 < 7$ so that the equation is true for $-4 \le x \le -3$. Finally, if $x > -3$ we have $|x+3|+|x+4| < 7 \Rightarrow x+3+x+4 < 7 \Rightarrow 2x+7 < 7 \Rightarrow x < 0$ so that $-3 < x < 0$. Combining the three inequalities the solution set is $-7 < x < 0$.

7. There are $5 \cdot 4 = 20$ ways for the person to enter and exit through different doors. The probability that the person uses an even door only upon entering is $\frac{2 \cdot 3}{20} = \frac{6}{20}$. The probability that the person uses an even door only upon exiting is $\frac{3 \cdot 2}{20} = \frac{6}{20}$. The probability that the person uses both an even door upon entering and exiting is $\frac{2 \cdot 1}{20} = \frac{2}{20}$. Therefore, the probability that at least one of the doors used is even is $\frac{6}{20}+\frac{6}{20}+\frac{2}{20} = \frac{14}{20} = \frac{7}{10}$.

8. For every 3 distinct digits selected from the set $\{1, 2, \ldots, 9\}$ there is exactly one way to arrange them into a number with increasing digits, and every number with increasing digits corresponds to one of these selections. Similarly, the numbers with decreasing digits correspond to the subsets with 3 elements of the set of all 10 digits. Hence, the answer is ${}_9C_3 + {}_{10}C_3 = \frac{9!}{6! \cdot 3!}+\frac{10!}{7! \cdot 3!} = \frac{9 \cdot 8 \cdot 7}{6}+\frac{10 \cdot 9 \cdot 8}{6} = 84+120 = 204$.